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NEARLY BIPARTITE GRAPHS WITH LARGE CHROMATIC NUMBER

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P. Erdős and A. Hajnal asked the following question. Does there exist a constant $\varepsilon>0$ with the following property: If every subgraph H of a graph G can be made bipartite by the omission of at most $\varepsilon|H|$ edges where $|H|$ denotes the number of vertices of H then $\chi(H)\leqq 3$.

The aim of this note is to give a negative answer to this question and consider the analogous problem for hypergraphs. The first was done also by L. Lovász who used a different construction.

0. Introduction

For a positive ε and a positive integer k we shall say that the graph G has the (ε, k)-vertex property $\big((\varepsilon, k)$-edge property resp.$\big)$ if

(i) $\chi(G)\geqq k$;

(ii) if H is a subgraph of G then H can be made bipartite by the omission of at most $\varepsilon|H|$ vertices ($\varepsilon|H|$ edges resp.). (Here $|H|$ denotes the number of vertices of H.)

Using this terminology, a question of *P.* Erdős and A. Hajnal can be formulated now as follows:

(*) Given ε and k, do there exist graphs with the (ε, k)-edge property?

Note that the (ε, k)-vertex property is clearly weaker than the (ε, k)-edge property. The Kneser graphs $K(n, k)$ ([4], cf. the next section) have the (ε, k)-vertex property for sufficiently large n. Further examples are the Borsuk graphs defined by Erdős and Hajnal [5] as follows. The vertex set is the k-dimensional unit sphere. Two points are adjacent iff their distance is greater than $2-\delta$ (where $\delta=\delta(\varepsilon)$ is a sufficiently small positive real). It is easy to see that any sufficiently dense finite subset of the sphere induces an appropriate *finite* subgraph. Still other examples were constructed by P. Erdős, A. Hajnal and E. Szemerédi [6]. They have shown that the graphs, the vertices of which are the r-tuples $\big(r>r(\varepsilon)\big)$ of elements of a sufficiently large linearly ordered set and two of them, $x_1<x_2<\ldots<x_r$ and $y_1<y_2<\ldots<y_r$ are adjacent if either $x_{i+1}=y_i$ or $y_{i+1}=x_i$ for $i=1, 2, \ldots, r-1$, have the (ε, k)-vertex property.

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Our note is divided into two parts. In the first part we give a construction which assigns to every graph G which has vertex transitive automorphism group and $\left(\frac{\varepsilon}{k-1}, k\right)$-vertex property a graph G^* having the (ε, k)-edge property. We also give another example of graphs having the (ε, k)-edge property. We shall use the graphs previously considered by A. Schrijver [11] in another context. A different example of graphs with the (ε, k)-edge-property was constructed by L. Lovász.

In the second part we show that the problem analogous to (*) for 3-graphs can be answered positively in a very strong sense. We show that for any sequence $\{k_n\}$ of positive integers such that $\lim_{n\to\infty} k_n = \infty$ there exists a 3-uniform hypergraph $F=(V, E)$ with

(i) $\chi(F)=\aleph_0$

(ii) Every subgraph H of F with $|H| \leqq k_n$ can be made bipartite by omitting at most n 3-tuples.

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1. Graphs which have the (ε, k)-edge-property

Construction 1.1. Let $G=(V, E)$ be a graph with $\chi(G)=k$ and let $<$ be an ordering of its vertices. Set $\{v_1, v_2, \dots, v_m\}$ where $v_i<v_j$ if $i<j$. We construct the graphs $G_1, G_2, \dots, G_m=G^*$ successively as follows:

(I) $G_1=G$.

(II) Suppose $G_i=(V_i, E_i)$, $1 \leqq i \leqq m-1$ has been constructed and V_i partitioned as $V_i = \bigcup_{j\leqq m} U_j^i$ such that $U_j^i=\{v_j\}$ for any $j>i$.

Denote by W_i the set of those vertices of G_i which are elements of $\bigcup_{j\leqq i} U_j^i$ and are adjacent to v_{i+1}. We define G_{i+1} as follows:

(α) if $|W_i| \leqq k-1$ set $G_i=G_{i+1}$;

(β) if $|W_i| \geqq k$ replace v_{i+1} by $\binom{|W_i|}{k-1}=t$ new points $u_1^{i+1}, \dots, u_t^{i+1}$ so that each $(k-1)$-subset of W_i is adjacent to one of them. Moreover if $\{v_{i+1}, v_{i'}\} \in E_i$ for $i'>i+1$, put $\{u_j^{i+1}, v_{i'}\} \in E_{i+1}$ for all $1 \leqq j \leqq t$. Set now

$$U_j^{i+1} = U_j^i \quad \text{for} \quad j \leqq i, \quad U_{i+1}^{i+1} = \{u_1^{i+1}, \dots, u_t^{i+1}\}$$

and

$$U_j^{i+1} = \{v_j\} \quad \text{for} \quad j > i+1.$$

Denote by F_{i+1} the set of all edges incident to vertices $u_1^{i+1}, \dots, u_t^{i+1}$ and put

$$E_{i+1} = E_i \cap [V-\{v_{i+1}\}]^2 \cup F_{i+1}.$$

Claim 1.2. $\chi(G^*)=k$.

Proof. Clearly $\chi(G^*) \leqq k$ as $\varphi\colon V^* \to V$ defined by $\varphi(U_j^m)=v_j$ for any j, $1 \leqq j \leqq m$ is a homomorphism. For the reverse inequality it suffices to prove that $\chi(G_{i+1})<k$ implies $\chi(G_i)<k$ for any $1 \leqq i \leqq m-1$. Suppose therefore that there is an r-coloring

c of G_{i+1} $(r<k)$. Now, no $(k-1)$-subset of W_i gets all the r colors, because one of the vertices $u_1^{i+1}, u_2^{i+1}, \ldots, u_t^{i+1}$ (say u_1^{i+1}) must be colored by a color different from the colors of points of W_i since it is connected to a $(k-1)$-subset of W_i which has all those colors. Define now a new graph

$$G'_{i+1} = \begin{cases} G_{i+1} + \text{all edges connecting vertices of } W_i \text{ to } u_1^{i+1} \text{ if } G_i \neq G_{i+1} \\ G_{i+1} \quad \text{if} \quad G_i = G_{i+1}. \end{cases}$$

Clearly c is also a coloring of G'_{i+1} and moreover G_i is a subgraph of G'_{i+1}. Thus $\chi(G_i) \leqq \chi(G'_{i+1}) \leqq r \leqq k$. ∎

Lemma 1.3. *Let $G=(V, E)$ be a graph with vertex transitive automorphism group which can be made bipartite by deletion of at most $\varepsilon|G|$ vertices. Then for any nonnegative real valued function $f\colon V \to \mathbf{R}^+$ there exists a set V' of vertices such that*
(i) *the subgraph of G induced on a set $V-V'$ is bipartite.*
(ii) $\sum_{v \in V'} f(v) \leqq \varepsilon \sum_{v \in V} f(v)$.

Proof. Let G and f be given. Suppose that G can be made bipartite by deleting the vertices $v_1, v_2, \ldots, v_p$; $p \leqq \varepsilon|G|$. It follows by vertex symmetry of G that

$$\sum_{1 \leqq i \leqq p} \sum_{\varphi \in \mathrm{Aut}\, G} f(\varphi(v_i)) = p \sum_{v \in V} f(v) \frac{|\mathrm{Aut}\, G|}{|G|}$$

and thus there exists $\varphi_0 \in \mathrm{Aut}\, G$ such that

$$\sum_{1 \leqq i \leqq p} f(\varphi_0(v_i)) \leqq p \frac{\sum_{v \in V} f(v)}{|G|} = \varepsilon \sum_{v \in V} f(v).$$

Set

$$V' = \{\varphi_0(v_1), \varphi_0(v_2), \ldots, \varphi_0(v_p)\}. \quad ∎$$

Let Z be a set with $2n+k$ elements. Consider the Kneser graph $K(n, k)$ the vertices of which are the n-element subsets of Z; two such sets are adjacent iff they are disjoint. It was conjectured by Kneser and proved by L. Lovász [8] and Bárány [1] that

$$\chi(K(n, k)) = k+2.$$

A few years ago L. Lovász told me that the following lemma holds.

Lemma 1.4. *For any $\varepsilon>0$ and positive integer k there exists n_0 such that for all positive integers $n \geqq n_0$ the Kneser graph $K=K(n, k)$ can be made bipartite by the omission of $\varepsilon|K|$ vertices.*

Proof. Split the $(2n+k)$-element set Z into two parts X, Y of cardinalities $|X| = \left\lceil \frac{2n+k}{2} \right\rceil$ and $|Y| = \left\lfloor \frac{2n+k}{2} \right\rfloor$, resp. Consider the set $\mathcal{B}$ of all n-tuples intersecting either X or Y in more than $\frac{n}{2}+\frac{k}{4}$ elements. Clearly, the subgraph of K induced on $\mathcal{B}$ is bipartite.

It can be shown that

$$\frac{|V(K)-\mathscr{B}|}{|K|}<\varepsilon \quad \text{if} \quad n \geqq n_0(k,\varepsilon) > \left(\frac{1}{\varepsilon}\right)^2\left(\frac{k}{2}+1\right)^2 \cdot \frac{4}{\pi} e^{1/3}. \quad \blacksquare$$

Let us take any ordering of the vertices of $K=K(n, k)$ and construct the graph $K^*=K^*(n, k)$ as in 1.1.

Then the following holds

Theorem 1.5. *For any $\varepsilon>0$ and any positive integer k the graphs $K^*=K^*(n, k)$, $n>n_0\left(k, \frac{\varepsilon}{k+1}\right)$ have the following properties:*

(i) $\chi(K^*)=k+2$.

(ii) *Any subgraph H of K^* can be made bipartite by deleting at most $\varepsilon|H|$ edges.*

Proof. Let ε and k be given. Set $\varepsilon'=\frac{\varepsilon}{k+1}$ and $n>n_0(k, \varepsilon')$. (i) follows immediately by Claim 1.2 and the result of Lovász [8]. We prove (ii). Let H be a subgraph of $K^*=K^*(n, k)$. Set $K=(V, E)$, $V=\{v_1, v_2, \dots, v_m\}$. Let $K^*=(V_m, E_m)$ where $V_m=\bigcup_{j\leqq m} U_j^m$ where the notation is taken from Construction 1.1. Consider the function

$$f\colon V \to \mathbf{R}^+ \quad \text{defined by} \quad f(v_j)=\frac{1}{2}\sum_{u\in V(H)\cap U_j} d(u), \quad 1 \leqq j \leqq m.$$

Then as K is clearly vertex symmetric, we obtain by Lemma 1.3 that K can be made bipartite omitting a subset $V'\subset V$ such that

$$\sum_{v\in V'} f(v) \leqq \frac{\varepsilon}{k+1}\sum_{v\in V} f(v) = \frac{\varepsilon}{k+1}|E(H)| \leqq \varepsilon|V(H)|.$$

Clearly omission of all edges incident to vertices $u\in U_j$ such that $v_j\in V'$ makes H bipartite. Moreover there are at most

$$\sum_{V_j\in V'}\sum_{u\in V_j\cap V(H)} d(u) = \sum_{v\in V'} f(v) \leqq \varepsilon|V(H)|$$

such edges. ∎

The graphs having the (ε, k)-edge property are clearly examples of graphs with large chromatic number not containing short cycles of odd length. I would like to know whether this can be generalized to avoid all short cycles.[1] For instance, do there exist graphs having the (ε, k)-vertex (or edge) property and girth greater than four?

Another way of constructing graphs for which (*) holds was indicated in [11] by A. Schrijver who described a class of color critical subgraphs of the Kneser graphs but did not examine how they are related to Conjecture (*). This will be done here.

We shall call an n-subset X' of $X=\{1, 2, \dots, 2n+k\}$ stable (cf. [6]) if it contains no pair of neighbours in the cyclic ordering of $\{1, 2, \dots, 2n+k\}$. In [11] A. Schrijver introduced the *reduced Kneser graph* $K'(n, k)=K'$ the vertices of which are the stable n-subsets of $\{1, 2, \dots, 2n+k\}$, two of them being adjacent iff they are disjoint. It is also proved in [11] that $\chi(K'(n, k))=k+2$.

[1] *Note added in proof:* The author has answered this question in the affirmative.

Now we shall show that the reduced Kneser graphs provide examples of graphs satisfying (*). Let k be fixed and n large. It can be seen quite easily that K' has $c_1 n^k(1+O(1))$ vertices and $c_2 n^k(1+O(1))$ edges, where c_1 and c_2 are constants depending on k only.

If e is an edge and v_1, v_2 its endpoints then denote by $v(e)$ the set of all $i\in\{1, 2, ..., 2n+k\}$ such that both $v_1\cap\{i, i+1\}$ and $v_2\cap\{i, i+1\}$ are nonempty (addition is mod $2n+k$). Clearly $|v(e)|\geqq 2n-k$.

Let H be a subgraph of $K'(n, k)$ with h edges. There exists an $i\in\{1, 2, ..., 2n+k\}$ such that

$$|\{e\in E(H);\ i\in v(e)\}| \geqq \frac{2n-k}{2n+k}h = \left(1-O\left(\frac{1}{n}\right)\right)h.$$

As $\{e\in E(H);\ i\in v(e)\}$ is clearly the edge set of a bipartite graph we obtain the following result:

Theorem 1.6. *Let H be a subgraph of $K'(n, k)$ the reduced Kneser graph. Set $h=|E(H)|$ and $m=|E(K'(n, k))|$. Then H can be made bipartite by deleting $c\dfrac{h}{m^{1/k}}$ edges (where c is a constant depending only on k).* ▮

We close this section by proving a stronger version of Conjecture (*).

Theorem 1.7. *For every $\varepsilon>0$ and k there exists a uniquely k-colorable graph H having the (ε, k)-edge-property.*

Proof. Set $H=K^*(n, k-1)\times K_k$ (where $\times$ denotes the (weak) direct product, cf. [8, p. 538]). Let n be so large that $K^*(n, k-1)$ has the $\left(\dfrac{\varepsilon}{k(k-1)}, k\right)$-edge-property. The unique colorability of H follows from [7] where it was proved that the direct product of a $(k+1)$-chromatic graph with K_k is uniquely $(k+1)$-colorable.

Let H' be a subgraph of H. Consider the projection $\pi\colon H'\to K^*(n, k-1)$ and assign to every edge e of $K^*(n, k-1)$ the number of edges $e'\in E(H')$ such that $\pi(e')=e$. Denote this number by $\alpha(e)$. Clearly $0\leqq\alpha(e)\leqq k(k-1)$ for every $e\in E(K^*(n, k-1))$. Consider the subgraph F of $K^*(n, k-1)$ formed by the edges e with $\alpha(e)>0$. F can be made bipartite by omitting $\dfrac{\varepsilon}{k(k-1)}|F|$ edges. Since $\alpha(e)\leqq k(k-1)$ for every $e\in E(F)$ it follows that H' can be made bipartite by omitting $\varepsilon|F|\leqq\varepsilon|H'|$ edges. ▮

2. Results concerning hypergraphs

In the previous section we have shown that for every $\varepsilon>0$ and positive integer k there exists a graph having the (ε, k)-edge property. Theorem 1.6 actually states a stronger result. It follows from this Theorem 1.6 that every subgraph H of $K'(n, k)$ having h edges can be made bipartite by omitting $ch^{1-1/k}$ edges. Other graphs with this property were constructed by L. Lovász. The question, whether $ch^{1-1/k}$ can be replaced by a slowlier growing function, remains open. Note that using similar examples this cannot be considerably improved and candidates for graphs which could improve Theorem 1.6 must be very asymmetrical. This follows from results proved in [10] where among others, the following has been noticed: Let $G=(V,E)$ be a k-chromatic ($k\geqq 5$) graph with vertex (edge) transitive automorphism group. Set $n=|V|$, $m=|E|$. Then G cannot be made bipartite by omission of less than $\sqrt{n}$ vertices $\left(\frac{m}{\sqrt{n}+1}\right.$ edges, resp.$\left.\right)$.

In contrast, the analogous question for hypergraphs has a positive answer in a very strong sense.

Theorem 2.1. *For every sequence $\{k_n\}$ of positive integers such that $\lim_{n\to\infty} k_n=\infty$ there exists a 3-uniform hypergraph $F=(V,E)$ such that*

(i) $\chi(F)=\aleph_0$;

(ii) *every subhypergraph H of F with $|H|\leqq k_n$ vertices can be made bipartite by the omission of at most n triples.*

Remark. One can prove in a similar way that the above Theorem is valid for r-uniform hypergraphs for any $r\geqq 3$.

In the proof of Theorem 2.1 we shall use the following lemma which we state here without proof because it can be done by standard methods due to P. Erdős (see e.g. [2]).

Lemma 2.2. *For any positive integers t, p and k there exists a system of graphs $G^j(t,k,p)=G^j=(V,E_j)$, $1\leqq j\leqq t$ such that*

(a) *(V,E) does not contain any cycles of length $\leqq k$, where $E=\bigcup_{1\leqq j\leqq t} E_j$.*

(b) *None of the G^j contains an independent set of more than $|V|/p$ vertices.* ∎

Proof of Theorem 2.1. We construct the sets of vertices and edges of the hypergraph F as unions of pairwise disjoint sets $V_1, V_2, \ldots$ and $E_1, E_2, \ldots$, resp.

Put $|V_1|=k_1$ and denote by G_1 the empty graph (the graph with no edges) with the vertex set V_1. Set $E_1=\emptyset$.

Suppose now that $V_1, V_2, \ldots, V_i$ and $E_1, E_2, \ldots, E_i$ have been constructed. Set $t_i=|V_1|+|V_2|+\ldots+|V_i|$ and $u_i=|E_1|+|E_2|+\ldots+|E_i|$.

Take a system of graphs $G^j_{i+1}=G^j(t_i, k_{u_i}, i+1)$ with the properties defined in Lemma 2.2. Put $V_{i+1}=V(G^1_{i+1})=\ldots=V(G^{t_i}_{i+1})$ and consider a one-to-one mapping

$$\varphi\colon \bigcup_{1\leqq j\leqq i} V_j \to \{G^1_{i+1}, \ldots, G^{t_i}_{i+1}\}.$$

Set now

$$E_{i+1}=\Big\{\{v\}\cup e\colon\ e\in E\big(\varphi(v)\big),\quad v\in \bigcup_{1\leqq j\leqq i} V_j\Big\}.$$

Let

$$V=\bigcup_{i=1}^{\infty} V_i \quad\text{and}\quad E=\bigcup_{i=1}^{\infty} E_i.$$

We show that $F=(V,E)$ has the required properties. Suppose first that $\chi(F)\leqq k$ for some finite k. Then there exist $i>i'\geqq k$ and nonempty monochromatic sets

$M_i \subset V_i$, $M_{i'} \subset V_{i'}$, $|M_i| \geq \frac{|V_i|}{k}$ which are colored by the same color. Hence, by property (b) of the graphs G_i^j and the construction of E_i there is a monochromatic triple $e \in E_i$ in $M_i \cup M_{i'}$, a contradiction, proving (i).

Let now W be a finite subset of V. Set $r_i = |[W]^3 \cap E_i|$ where $[W]^3$ denotes the set of triples from W. If $r_{i+1} \leq k_{u_i}$ for every $i = 1, 2, \dots$ then $[W]^2 \cap E_{i+1}^*$ is a forest for any i, where $E_{i+1}^* = \bigcup_{1 \leq j \leq t_i} E(G_{i+1}^j)$. This implies that the hypergraph induced on W is bipartite and there is nothing to prove. Take therefore the largest i (if exists) such that $r_{i+1} > k_{u_i}$. Then $|W| > k_{u_i}$ but the subhypergraph of E induced on W can be made bipartite by omitting the (at most u_i) edges of $\bigcup_{j=1}^{i} E_j \cap [W]^3$. ∎

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